

On α^{++} -Stable Graphs

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February 1, 2008

Abstract

The *stability number* of the graph G , denoted by $\alpha(G)$, is the cardinality of a maximum stable set of G . A graph is well-covered if every maximal stable set has the same size. G is a *König-Egerváry graph* if its order equals $\alpha(G) + \mu(G)$, where $\mu(G)$ is the cardinality of a maximum matching in G . In this paper we characterize α^{++} -stable graphs, namely, the graphs whose stability numbers are invariant to adding any two edges from their complements. We show that a König-Egerváry graph is α^{++} -stable if and only if it has a perfect matching consisting of pendant edges and no four vertices of the graph span a cycle. As a corollary it gives necessary and sufficient conditions for α^{++} -stability of bipartite graphs and trees. For instance, we prove that a bipartite graph is α^{++} -stable if and only if it is well-covered and C_4 -free.

1 Introduction

All the graphs considered in this paper have at least two vertices. For such a graph $G = (V, E)$ we denote its vertex set by $V = V(G)$ and its edge set by $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. By $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F , for $F \subset E(G)$, and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \cup\{N(v) : v \in A\}$, for $A \subset V$. If $|N(v)| = |\{w\}| = 1$, then v is a *pendant vertex* and vw is a *pendant edge* of G . By C_n, K_n, P_n we denote the chordless cycle on $n \geq 4$ vertices, the complete graph on $n \geq 2$ vertices, and respectively the chordless path on $n \geq 3$ vertices.

A stable set of maximum size will be referred as to a *maximum stable set* of G . The *stability number* of G , denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G . Let $\Omega(G)$ denotes $\{S : S \text{ is a maximum stable set of } G\}$ and

$\xi(G) = |\cap\{S : S \in \Omega(G)\}|$. We call $\{\Omega_1, \Omega_2\}$ a cover of $\Omega(G)$ if $\Omega_1, \Omega_2 \subset \Omega(G)$ and $\Omega_1 \cup \Omega_2 = \Omega(G)$; by $\xi(\Omega_i), i = 1, 2$, we mean the number $|\cap\{S : S \in \Omega(G_i)\}|$.

A *matching* is a set of non-incident edges of G ; a matching of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is a matching covering all the vertices of G . G is a *König-Egerváry graph* provided $\alpha(G) + \mu(G) = |V(G)|$, [2], [15].

A graph G is α^+ -stable if $\alpha(G + e) = \alpha(G)$, for any edge $e \in E(\overline{G})$, where \overline{G} is the complement of G , [7]. Haynes et al. have characterized the α^+ -stable as follows:

Theorem 1.1 [8] *A graph G is α^+ -stable if and only if $\xi(G) \leq 1$.*

Theorem 1.1 implies that for an α^+ -stable graph either $\xi(G) = 0$ or $\xi(G) = 1$. This motivates the following definition. A graph G is called (i) α_0^+ -stable whenever $\xi(G) = 0$, and (ii) α_1^+ -stable provided $\xi(G) = 1$, [10]. For instance, C_4 is α_0^+ -stable, while the graphs $K_3 + e, K_4 + e$ in Figure 1 are α_1^+ -stable.

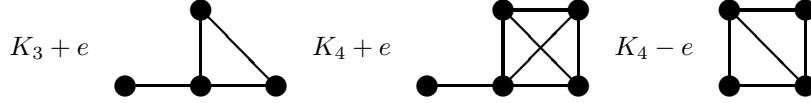


Figure 1: $K_3 + e$ and $K_4 + e$ are α_1^+ -stable graphs; $K_4 - e$ is not α^+ -stable.

In [8] it was shown that an α^+ -stable tree can be only α_0^+ -stable, and this is exactly the case of trees possessing a perfect matching. This result was generalized to bipartite graphs in [10]. Nevertheless, there exist both α_1^+ -stable König-Egerváry graphs (e.g., the graph $K_3 + e$ in Figure 1), and α_0^+ -stable König-Egerváry graphs (e.g., all α^+ -stable bipartite graphs). A necessary (but not sufficient, e.g., $K_4 - e$) condition for α^+ -stability is:

Proposition 1.2 [12] *Any α^+ -stable König-Egerváry graph has a perfect matching.*

Let define a graph G as α^{++} -stable whenever $\alpha(G + e_1 + e_2) = \alpha(G)$, for any $e_1, e_2 \in E(\overline{G})$, and $\alpha_{P_3}^+$ -stable provided $\alpha(G + e_1 + e_2) = \alpha(G)$, for any $e_1, e_2 \in E(\overline{G})$ that have a common endpoint. Gunther et al., [7], studied the structure of α^+ -stable trees, and in [10], [12], some of their results were extended to bipartite graphs and König-Egerváry graphs.

In this paper we characterize α^{++} -stable graphs. We settle a number of connections between α^+ -stable graphs, $\alpha_{P_3}^+$ -stable and α^{++} -stable graphs. In particular, we show that any α_0^+ -stable graph is $\alpha_{P_3}^+$ -stable. We also give a necessary and sufficient condition for a graph to be α_1^+ -stable and $\alpha_{P_3}^+$ -stable at the same time.

We prove that a König-Egerváry graph is α^{++} -stable if and only if it has a perfect matching consisting of only pendant edges and contains no cycle on 4 vertices. Using this result we describe α^{++} -stable bipartite graphs and α^{++} -stable trees. For instance, it is shown that a bipartite graph is α^{++} -stable if and only if it is well-covered and C_4 -free.

2 α^{++} -stable graphs

Notice that any α^{++} -stable graph is $\alpha_{P_3}^+$ -stable, but the converse is not generally true. For instance, C_4 is $\alpha_{P_3}^+$ -stable but not α^{++} -stable. Let us also observe that for $n \geq 2$, the graph $K_n - e$ is $\alpha_{P_3}^+$ -stable, but it is not α^+ -stable.

Proposition 2.1 *If $G \neq K_n - e$, $n \geq 2$ is $\alpha_{P_3}^+$ -stable, then it is also α^+ -stable.*

Proof. Assume that $\alpha(G) = 2$. Since $G \neq K_n - e$, it follows that $|\Omega(G)| \geq 2$, and consequently G is α^+ -stable, as well. For $\alpha(G) \geq 3$, suppose, on the contrary, that G is not α^+ -stable, and let $x, y \in \cap\{S : S \in \Omega(G)\}$. Hence, for $S \in \Omega(G)$ and $z \in S - \{x, y\}$, we obtain that $\alpha(G + xy + xz) < \alpha(G)$, in contradiction with the fact that G is $\alpha_{P_3}^+$ -stable. Therefore, G must be α^+ -stable. ■

It is worth observing that an α^+ -stable graph is not necessarily $\alpha_{P_3}^+$ -stable. For instance, $K_3 + e$ is α^+ -stable, in fact it is α_1^+ -stable, but it is not $\alpha_{P_3}^+$ -stable. However, there exist graphs that are both α_1^+ -stable and $\alpha_{P_3}^+$ -stable; e.g., the graph $K_4 + e$ in Figure 1.

Proposition 2.2 *Any α_0^+ -stable graph is also $\alpha_{P_3}^+$ -stable.*

Proof. Suppose, on the contrary, that some α_0^+ -stable graph G is not $\alpha_{P_3}^+$ -stable, i.e., there are $x, y, z \in V$ such that $\alpha(G + xy + xz) < \alpha(G)$. Hence, it follows that $x \in \cap\{S : S \in \Omega(G)\}$, in contradiction with the fact that G is α_0^+ -stable. ■

Combining Theorem 1.1 and Propositions 2.1, 2.2, we get:

Corollary 2.3 *If $G \neq K_n - e$ has $\alpha(G) = 2$, then G is:*

- (i) α^+ -stable if and only if $|\Omega(G)| \geq 2$;
- (ii) $\alpha_{P_3}^+$ -stable if and only if either it is α_0^+ -stable or it is α_1^+ -stable and $|\Omega(G)| \geq 3$;
- (iii) α^{++} -stable if and only if $|\Omega(G)| \geq 3$;
- (iv) $\alpha_{P_3}^+$ -stable if and only if it is α^+ -stable and $G \neq P_3(K_1, K_m, K_2)$, where $P_3(K_1, K_m, K_2)$ is the graph obtained by substituting the vertices of P_3 respectively, by K_1, K_m, K_2 , and joining all the vertices of K_m with the two vertices of K_2 and the single vertex of K_1 .

Proposition 2.4 *If $G = (V, E)$ has $\alpha(G) \geq 3$, then the following assertions are equivalent:*

- (i) G is α^+ -stable;
- (ii) either G is $\alpha_{P_3}^+$ -stable, or there exist three vertices $x, y, z \in V$ such that $|\{x, y\} \cap S| \cdot |\{x, z\} \cap S| \geq 2$ holds for any $S \in \Omega(G)$, and x is the unique vertex of G with this property.

Proof. (i) \Rightarrow (ii) Let G be α^+ -stable, but not $\alpha_{P_3}^+$ -stable. Hence, there are $x, y, z \in V$ such that $\alpha(G + xy + xz) < \alpha(G)$. Therefore, we get that $x \in \cap\{S : S \in \Omega(G)\}$, because otherwise, any $S \in \Omega(G)$ not containing x is still stable in $G + xy + xz$, and

consequently, we obtain $\alpha(G + xy + xz) = \alpha(G)$, in contradiction with the assumption on G . In addition, each $S \in \Omega(G)$ satisfies $|S \cap \{y, z\}| \geq 1$, since otherwise, if some $S_0 \in \Omega(G)$ has $S_0 \cap \{y, z\} = \emptyset$, then S_0 is stable in $G + xy + xz$ and this yields $\alpha(G + xy + xz) = \alpha(G)$, again in contradiction with the assumption on G . Finally, x is unique, because otherwise $\xi(G) \geq 2$, which contradicts the α^+ -stability of G .

(ii) \Rightarrow (i) If G is $\alpha_{P_3}^+$ -stable and $\alpha(G) \geq 3$, then by Proposition 2.1, G is α^+ -stable. Further, if there are $x, y, z \in V$ such that $|\{x, y\} \cap S| \cdot |\{x, z\} \cap S| \geq 2$ holds for any $S \in \Omega(G)$, and x is unique with this property, it follows that G is not $\alpha_{P_3}^+$ -stable, because $\alpha(G + xy + xz) < \alpha(G)$, but it is α_1^+ -stable, since $\{x\} = \cap\{S : S \in \Omega(G)\}$. ■

Lemma 2.5 *If for any $x, y \in V(G)$ there exists $S \in \Omega(G)$ such that $x, y \in V(G) - S$, then G is both α_0^+ -stable and α^{++} -stable.*

Proof. Suppose, on the contrary, that there exists $x \in \cap\{S : S \in \Omega(G)\}$. Then, for any $y \in V(G) - \{x\}$ and $S \in \Omega(G)$, we get $\{x, y\} \cap S \neq \emptyset$, in contradiction with the premises on G . Therefore, G is α_0^+ -stable. According to Proposition 2.2, it follows that G is $\alpha_{P_3}^+$ -stable, too. Assume that G is not α^{++} -stable. Hence, since G is $\alpha_{P_3}^+$ -stable, we infer that there are $x, y, u, v \in V(G)$, pairwise distinct, such that $\alpha(G + xy + uv) < \alpha(G)$. Let $S \in \Omega(G)$ be such that $x, u \in V(G) - S$. Then S is stable in $G + xy + uv$, in contradiction with the assumption on G . Consequently, G is α^{++} -stable. ■

As an example, $C_{2k+1}, k \geq 2$ and $K_n, n \geq 3$ are both α_0^+ -stable and α^{++} -stable, according to Lemma 2.5. Notice that the converse of Lemma 2.5 is not generally true; see, for instance, the graphs $C_{2k}, k \geq 3$. There exist α_0^+ -stable graphs that are not α^{++} -stable (e.g., C_4), and vice-versa, there are α^{++} -stable that are not α_0^+ -stable (e.g., $K_4 + e$).

Proposition 2.6 *Let G be α_1^+ -stable, $\{v\} = \cap\{S : S \in \Omega(G)\}$ and $G_0 = G - N[v]$. If G is not $\alpha_{P_3}^+$ -stable, then there are x and y belonging to the same connected component of G_0 , such that $\alpha(G + xv + yv) < \alpha(G)$. In other words, there exists a path connecting x and y , which avoid the neighborhood of v .*

Proof. Let $\{H_k : 1 \leq k \leq s\}, s \geq 2$, be the connected components of G_0 , and suppose that there are x and y belonging to different connected components of G_0 , (say respectively H_i, H_j), such that $\alpha(G + xv + yv) < \alpha(G)$. Since G_0 is α_0^+ -stable, it follows that all its connected components are α_0^+ -stable, as well. Let $S_k \in \Omega(H_k), 1 \leq k \leq s$, and $S_i \in \Omega(H_i), S_j \in \Omega(H_j)$ be such that $x \notin S_i, y \notin S_j$, which exist, because all H_k are α_0^+ -stable. Hence, we get that $\{v\} \cup (\cup\{S_k : 1 \leq k \leq s\})$ is stable in $G + xv + yv$, in contradiction with $\alpha(G + xv + yv) < \alpha(G)$. ■

Theorem 2.7 *Let G be α_1^+ -stable, $\{v\} = \cap\{S : S \in \Omega(G)\}$ and $G_0 = G - N[v]$. Then G is $\alpha_{P_3}^+$ -stable if and only if for every pair $x, y \in V(G_0)$ there exists $S_0 \in \Omega(G_0)$ such that $x, y \in V(G_0) - S_0$.*

Proof. Let $x, y \in V(G_0)$. By definition of G_0 , it follows that $x, y \notin N[v]$, and since G is $\alpha_{P_3}^+$ -stable, we get that $\alpha(G + xv + yv) = \alpha(G)$. Therefore, there is $S \in \Omega(G)$ such that $x, y \in V(G) - S$. Hence, $x, y \in V(G_0) - S_0$, where $S_0 = S - \{v\} \in \Omega(G_0)$.

Conversely, G is α_1^+ -stable, and for every pair $x, y \in V(G_0)$ there exists $S \in \Omega(G_0)$ with $x, y \in V(G_0) - S_0$. Assume that G is not $\alpha_{P_3}^+$ -stable. Hence, there are $x, y \in V(G)$ such that $\alpha(G + xv + yv) < \alpha(G)$, since G is α_1^+ -stable. Let $S_0 \in \Omega(G_0)$ be such that $x, y \in V(G_0) - S_0$. Then, it follows that $S_0 \cup \{v\} \in \Omega(G)$, in contradiction with the assumption on G . Therefore, G is also $\alpha_{P_3}^+$ -stable. ■

Proposition 2.8 *A graph G is not $\alpha_{P_3}^+$ -stable if and only if $\xi(G) \geq 1$ and there exists a cover $\{\Omega_1, \Omega_2\}$ of $\Omega(G)$, such that $\xi(\Omega_i) \geq 2, i = 1, 2$.*

Proof. If G is not $\alpha_{P_3}^+$ -stable, then $\alpha(G + e_1 + e_2) < \alpha(G)$ holds for some $e_1, e_2 \in E(\overline{G})$ that have a common endpoint. Suppose $e_1 = xy, e_2 = yz$. Let us define

$$\Omega_1 = \{S : x, y \in S \in \Omega(G)\} \text{ and } \Omega_2 = \{S : y, z \in S \in \Omega(G)\}.$$

Hence, it follows that $\xi(G) \geq 1$ and $\xi(\Omega_i) \geq 2, i = 1, 2$.

Conversely, assume that $\xi(G) \geq 1$, i.e., there exists at least one vertex belonging to $\cap\{S : S \in \Omega(G)\}$, say y , and that there is some cover $\{\Omega_1, \Omega_2\}$ of $\Omega(G)$, such that $\xi(\Omega_i) \geq 2, i = 1, 2$. If $x \in \cap\{S : S \in \Omega_1\} - \{y\}$ and $v \in \cap\{S : S \in \Omega_2\} - \{y\}$, then $\alpha(G + xy + uv) < \alpha(G)$, because any $S \in \Omega(G)$ contains at least one of the pairs $\{x, y\}$ or $\{y, v\}$. Therefore, G can not be $\alpha_{P_3}^+$ -stable. ■

Proposition 2.9 *A graph G is not α^{++} -stable if and only if there exists a cover $\{\Omega_1, \Omega_2\}$ of $\Omega(G)$, such that $\xi(\Omega_i) \geq 2, i = 1, 2$.*

Proof. If G is not an α^{++} -stable graph, then $\alpha(G + e_1 + e_2) < \alpha(G)$ holds for some $e_1, e_2 \in E(\overline{G})$. Suppose $e_1 = xy, e_2 = uv$. Let us define $\Omega_1 = \{S : x, y \in S \in \Omega(G)\}$ and $\Omega_2 = \{S : u, v \in S \in \Omega(G)\}$. Hence, it follows that $\xi(\Omega_i) \geq 2, i = 1, 2$. Suppose that there exists $S \in \Omega - (\Omega_1 \cup \Omega_2)$. Then $S \in \Omega(G + e_1 + e_2)$, that contradicts the inequality $\alpha(G + e_1 + e_2) < \alpha(G)$. Hence, $\Omega_1 \cup \Omega_2 = \Omega$, which means that $\{\Omega_1, \Omega_2\}$ is a cover we were supposed to find.

Conversely, assume that there is a cover $\{\Omega_1, \Omega_2\}$ of $\Omega(G)$ with $\xi(\Omega_i) \geq 2, i = 1, 2$. If $x, y \in \cap\{S : S \in \Omega_1\}$ and $u, v \in \cap\{S : S \in \Omega_2\}$, then $\alpha(G + xy + uv) < \alpha(G)$, because any $S \in \Omega(G)$ contains at least one of the pairs $\{x, y\}$ or $\{u, v\}$. Therefore, G is not α^{++} -stable. ■

Combining Propositions 2.4 and 2.9, we deduce the following:

Theorem 2.10 *For a graph G the following assertions are equivalent:*

- (i) G is α^{++} -stable;
- (ii) G is α^+ -stable and $\Omega(G + e_1) \cap \Omega(G + e_1) \neq \emptyset$ for any $e_1, e_2 \in E(\overline{G})$;
- (iii) $\Omega(G) \cap \Omega(G + e_1) \cap \Omega(G + e_1) \neq \emptyset$ for any $e_1, e_2 \in E(\overline{G})$;
- (iv) G is α^+ -stable and $|\cap\{S : S \in \Omega(G + e)\}| \leq 1$ for any $e \in E(\overline{G})$;
- (v) G is $\alpha_{P_3}^+$ -stable and there are no $e_1, e_2 \in E(\overline{G}), e_1 = xy, e_2 = uv$, such that:

$$\{x, y\} \cap \{u, v\} = \emptyset, \text{ and for any } S \in \Omega(G), \max\{|S \cap \{x, y\}|, |S \cap \{u, v\}|\} = 2;$$

- (vi) for any cover $\{\Omega_1, \Omega_2\}$ of $\Omega(G)$ either $\xi(\Omega_1) \leq 1$ or $\xi(\Omega_2) \leq 1$ holds.

3 α^{++} -stable König-Egerváry graphs

According to a well-known result of König, [9], and Egerváry, [4], any bipartite graph is a König-Egerváry graph. This class includes also non-bipartite graphs (see, for instance, the graph $K_3 + e$ in Figure 1). If $G_i = (V_i, E_i), i = 1, 2$, are two disjoint graphs, then $G = G_1 * G_2$ is defined as the graph with $V(G) = V(G_1) \cup V(G_2)$, and

$$E(G) = E(G_1) \cup E(G_2) \cup \{xy : \text{for some } x \in V(G_1) \text{ and } y \in V(G_2)\}.$$

Proposition 3.1 [12] *The following assertions are equivalent:*

- (i) G is a König-Egerváry graph;
- (ii) $G = H_1 * H_2$, where $V(H_1) = S \in \Omega(G)$ and $|V(H_1)| \geq \mu(G) = |V(H_2)|$;
- (iii) $G = H_1 * H_2$, where $V(H_1) = S$ is a stable set in G , $|S| \geq |V(H_2)|$ and $(S, V(H_2))$ contains a matching M with $|M| = |V(H_2)|$.

It is easy to see that a König-Egerváry graph G has a perfect matching if and only if $\alpha(G) = \mu(G)$. The edges of any maximum matching of a König-Egerváry graph have a specific position with respect to the maximum stable sets.

Lemma 3.2 [12] *If G is a König-Egerváry graph, then for any $S \in \Omega(G)$ every maximum matching of G is contained in $(S, V(G) - S)$.*

Proposition 3.3 *If M is a maximum matching in a graph G and H is a subgraph of G such that $M = (M \cap E(H)) \cup (M \cap E(G - H))$, then $\mu(G) = \mu(H) + \mu(G - H)$.*

Proof. Clearly, $M \cap E(H)$ and $M \cap E(G - H)$ are matchings in H and $G - H$, respectively. Let M_1, M_2 be maximum matchings in H and $G - H$, respectively. If $\mu(H) = |M_1| > |M \cap E(H)|$, or $\mu(G - H) = |M_2| > |M \cap E(G - H)|$, then $M_1 \cup M_2$ is a matching in G of cardinality larger than $|M|$, in contradiction with $|M| = \mu(G)$. Therefore, $\mu(G) = \mu(H) + \mu(G - H)$. ■

Proposition 3.4 *If M is a maximum matching in a König-Egerváry graph G , and H is a subgraph of G such that $M = (M \cap E(H)) \cup (M \cap E(G - H))$, then*

- (i) H and $G - H$ are König-Egerváry graphs;
- (ii) $\alpha(G) = \alpha(H) + \alpha(G - H)$.

Proof. Let $S \in \Omega(G)$, $S_1 = S \cap V(H)$ and $S_2 = S \cap V(G - H)$. By Lemma 3.2, $M \subseteq (S, V(G) - S)$, and according to Proposition 3.1(ii), $G = H_1 * H_2$, where $V(H_1) = S \in \Omega(G)$ and $|V(H_1)| \geq \mu(G) = |M| = |V(H_2)|$. Hence, we infer that: $V(H) = S_1 \cup (V(H_2) - V(G - H))$, S_1 is stable in H , $M \cap E(H)$ is a matching in H of size $|V(H_2) - V(G - H)|$, and $|S_1| \geq |V(H_2) - V(G - H)|$, i.e., H is a König-Egerváry graph, according to Proposition 3.1(iii). Similarly, $G - H$ is also a König-Egerváry graph. Since, by Proposition 3.3, $\mu(G) = \mu(H) + \mu(G - H)$ and all $G, H, G - H$ are König-Egerváry graphs, we may conclude that $\alpha(G) = \alpha(H) + \alpha(G - H)$. ■

Lemma 3.5 *If H is a subgraph of G , such that $\alpha(G) = \alpha(H) + \alpha(G - H)$ and G is α^{++} -stable, then H is α^{++} -stable, as well.*

Proof. Since $\alpha(G) = \alpha(H) + \alpha(G - H)$, it follows that any $S \in \Omega(G)$ satisfies $|S \cap V(H)| = \alpha(H)$. So, if $\alpha(H + e_1 + e_2) < \alpha(H)$, for some $e_1, e_2 \in E(\overline{H})$, it follows that $\alpha(G + e_1 + e_2) < \alpha(G)$, as well. ■

Lemma 3.6 *If G is of order 6, has a Hamiltonian path and $\alpha(G) = 3$, then G is not α^{++} -stable.*

Proof. Suppose that $V(G) = \{v_i : 1 \leq i \leq 6\}$ and $(v_i, v_{i+1}) \in E(G)$ for any $i \in \{1, \dots, 5\}$. Then $H = G + v_1v_3 + v_4v_6$ has $\alpha(H) = 2$, i.e., G is not α^{++} -stable. ■

Proposition 3.7 *If $G \neq K_n - e, n = 2, 3$ is an α^{++} -stable König-Egerváry graph, then G has a perfect matching consisting of only pendant edges.*

Proof. If G is α^{++} -stable then, clearly, G is $\alpha_{P_3}^+$ -stable too. Hence, by Proposition 2.1 if $G \neq K_n - e$ then it is also α^+ -stable. It is not difficult to check that $K_n - e$ can be a König-Egerváry graph only for $n = 2, 3$. Therefore, if a König-Egerváry graph $G \neq K_n - e, n = 2, 3$ is α^{++} -stable, then it is also α^+ -stable. Now Proposition 1.2 ensures that G has a perfect matching, say $M = \{a_i b_i : 1 \leq i \leq \alpha(G)\}$. According to Proposition 3.1 and Lemma 3.2, we may assume that $S = \{a_i : 1 \leq i \leq \alpha(G)\} \in \Omega(G)$. We show that M consists of only pendant edges. Suppose, on the contrary, that some $a_k b_k \in M$ is not pendant.

Case 1. There exists some b_i such that $a_k b_i, b_i b_k \in E(G)$ (see Figure 2(a)). If $H = G[\{a_k, b_i, a_i, b_k\}]$, then Proposition 3.4(ii) implies that $\alpha(G) = \alpha(H) + \alpha(G - H)$. Since H is not α^{++} -stable, it follows, by Lemma 3.5, that G could not be α^{++} -stable, in contradiction with the premises on G .

Case 2. There exist $a_i b_i \in M$ with $a_k b_i, a_i b_k \in E(G)$ (see Figure 2(b)). If $H = G[\{a_k, b_i, a_i, b_k\}]$, then Proposition 3.4(ii) ensures that $\alpha(G) = \alpha(H) + \alpha(G - H)$. Since H is not α^{++} -stable, it follows, by Lemma 3.5, that G could not be α^{++} -stable, in contradiction with the premises on G .

Case 3. There exist a_i, b_i, a_j, b_j , such that $a_i b_i, a_j b_j \in M$ and $a_k b_i, a_j b_k \in E(G)$. In addition, we can assume that $b_i b_k, b_k b_j \notin E(G)$, otherwise we return to *Case 1*. Hence, $H = G[\{a_i, a_k, a_j, b_i, b_k, b_j\}]$ contains a path on 6 vertices (see Figure 2(c)). Since $\alpha(H) = |\{a_i, a_k, a_j\}| = 3$, Lemma 3.6 implies that H is not α^{++} -stable, and because $\alpha(G) = \alpha(H) + \alpha(G - H)$ is true according to Proposition 3.4(ii), we get, by Lemma 3.5, that G cannot be α^{++} -stable, in contradiction with the premises on G .

Thus, M must consist of only pendant edges. ■

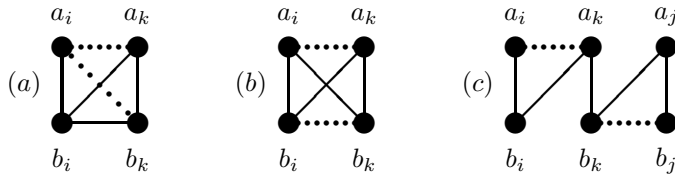


Figure 2: Non- α^{++} -stable König-Egerváry graphs.

It is worth observing that : (a) Proposition 3.7 fails for non-König-Egerváry graphs; e.g., C_5 is α^{++} -stable and has no perfect matching; (b) the converse of

Proposition 3.7, within the class of König-Egerváry graphs, is not generally true. For instance, the graph G_1 in Figure 3 has a perfect matching consisting of only pendant edges and it is not α^{++} -stable (because $\alpha(G_1 + ad + bc) < \alpha(G_1)$), while the graph G_2 in the same figure has a perfect matching consisting of only pendant edges, and it is also α^{++} -stable.

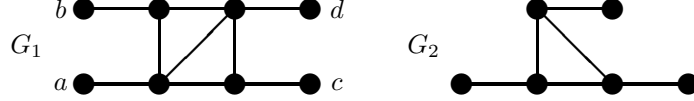


Figure 3: König-Egerváry graphs with a perfect matching consisting of pendant edges.

However, we can show that:

Proposition 3.8 *Any graph that has a perfect matching consisting of only pendant edges is $\alpha_{P_3}^+$ -stable.*

Proof. Let G be a graph that has a perfect matching $M = \{a_i b_i : 1 \leq i \leq \mu(G)\}$, consisting of only pendant edges, and suppose that $S_0 = \{a_i : 1 \leq i \leq \mu(G)\} \in \Omega(G)$. Let denote $H = G + e_1 + e_2$, where $e_1, e_2 \in E(\overline{G})$ are such that they have a common endpoint, say $e_1 = uv, e_2 = vw$. We distinguish between the following cases:

Case 1. If $u, v, w \notin S_0$, then $S_0 \in \Omega(H)$, and $\alpha(H) = \alpha(G)$.

Case 2. If $u, v \notin S_0$ or $u, w \notin S_0$, then $S_0 \in \Omega(H)$, and $\alpha(H) = \alpha(G)$.

Case 3. If $u \notin S_0$ and $v = a_i, w = a_j$, then $S_0 \cup \{b_j\} - \{a_j\} \in \Omega(H)$, and $\alpha(H) = \alpha(G)$.

Case 4. If $v \notin S_0$ and $u = a_i, w = a_j$, then $S_0 \in \Omega(H)$, and $\alpha(H) = \alpha(G)$.

Case 5. If $u = a_i, v = a_j, w = a_k$, then $S_0 \cup \{b_j\} - \{a_j\} \in \Omega(H)$, and $\alpha(H) = \alpha(G)$.

Consequently, G is $\alpha_{P_3}^+$ -stable. ■

Theorem 3.9 *A graph G that has a perfect matching consisting of only pendant edges is α^{++} -stable if and only if G contains no cycle on 4 vertices.*

Proof. Let $M = \{a_i b_i : 1 \leq i \leq \mu(G)\}$ be the perfect matching of G . Without loss of generality, we can assume that $S_0 = \{a_i : 1 \leq i \leq \mu(G)\} \in \Omega(G)$.

Suppose, on the contrary, that there is

$$D = \{b_i, b_j, b_k, b_m\} \text{ with } b_i b_j, b_j b_k, b_k b_m, b_m b_i \in E(G),$$

i.e., $G[D]$ contains a Hamiltonian cycle. If $H = G[\{a_i, a_j, a_k, a_m\} \cup D]$, then $\alpha(G) = \alpha(G - H) + \alpha(H)$, since any $S \in \Omega(G)$ satisfies $|S \cap \{a_q, b_q\}| = 1$ for each $a_q b_q \in M$. On the one hand, by Lemma 3.5, H should be α^{++} -stable. On the other hand, $\alpha(H + a_i a_k + a_j a_m) = 3 < \alpha(H) = 4$, which brings a contradiction. Therefore, G has no cycle on 4 vertices.

Conversely, let G be such that no 4 vertices span a cycle. Assume, on the contrary, that G is not α^{++} -stable, i.e., there are $e_1, e_2 \in E(\overline{G})$ such that $H = G + e_1 + e_2$ has $\alpha(H) < \alpha(G)$. If at least one of e_1, e_2 joins two vertices from $\{b_i : 1 \leq i \leq \mu(G)\}$, or one from $\{a_i : 1 \leq i \leq \mu(G)\}$ and the second from $\{b_i : 1 \leq i \leq \mu(G)\}$, then

$\alpha(H) = \alpha(G)$. According to Proposition 3.8, the same result follows if e_1, e_2 have a common endpoint. Suppose that $e_1 = a_i a_j, e_2 = a_k a_m$ and a_i, a_j, a_k, a_m are pairwise distinct. Hence, we get:

- $b_i b_k \in E(G)$, otherwise any $S \in \Omega(G)$ containing $\{a_j, a_m, b_i, b_k\}$ is stable in H ;
- $b_j b_m \in E(G)$, otherwise any $S \in \Omega(G)$ containing $\{a_i, a_k, b_j, b_m\}$ is stable in H ;
- $b_i b_m \in E(G)$, otherwise any $S \in \Omega(G)$ containing $\{a_j, a_k, b_i, b_m\}$ is stable in H ;
- $b_j b_k \in E(G)$, otherwise any $S \in \Omega(G)$ containing $\{a_i, a_m, b_j, b_k\}$ is stable in H .

It follows that $b_i b_k, b_j b_k, b_j b_m, b_i b_m \in E(G)$, i.e., $\{b_i, b_j, b_k, b_m\}$ spans a 4-cycle in G , in contradiction with the premises on G . Consequently, G is α^{++} -stable. ■

Combining Proposition 3.7 and Theorem 3.9, we obtain the following characterization of α^{++} -stable König-Egerváry graphs.

Theorem 3.10 *A König-Egerváry graph is α^{++} -stable if and only if it has a perfect matching consisting of only pendant edges and contains no cycle on 4 vertices.*

Recall that a graph G is called: (a) *well-covered* if every maximal stable set of G is also a maximum stable set, i.e., it is in $\Omega(G)$, [13]; (b) *very well-covered* provided G is well-covered and $|V(G)| = 2\alpha(G)$, [5]. The following result extends the characterization that Finbow, Hartnell and Nowakowski give in [6] for well-covered graphs having the girth ≥ 6 .

Proposition 3.11 *Let G be a graph of girth ≥ 6 , which is isomorphic to neither C_7 nor K_1 . Then the following assertions are equivalent:*

- (i) G is well-covered;
- (ii) G has a perfect matching consisting of pendant edges;
- (iii) G is very well-covered;
- (iv) G is a König-Egerváry α_0^+ -stable graph with exactly $\alpha(G)$ pendant vertices;
- (v) G is a König-Egerváry α^{++} -stable graph.

Proof. The equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) are done in [6]. In [11] it has been proved that (iii) \Leftrightarrow (iv). Finally, (ii) \Leftrightarrow (v) is true by Theorem 3.10. ■

Corollary 3.12 *For a bipartite graph G the following assertions are equivalent:*

- (i) G is α^{++} -stable;
- (ii) G is C_4 -free and has a perfect matching consisting of only pendant edges;
- (iii) G is C_4 -free and well-covered.

Corollary 3.13 C_n is α^{++} -stable if and only if n is odd.

Proof. For any $n \geq 2$, C_{2n} is not an α^{++} -stable graph according to Corollary 3.12.

Assume, on the contrary, that C_{2n+1} is not an α^{++} -stable graph. Hence, there are $e_1, e_2 \in E(\overline{C_{2n+1}})$, $e_1 = xy, e_2 = uv$ such that $\alpha(C_{2n+1} + e_1 + e_2) < \alpha(C_{2n+1})$. We may suppose, without loss of generality, that $x = v_1 \neq u$. Now, if $u = v_{2i+1}$ (for some $i \neq 0$), then $x, u \notin S = \{v_{2i} : 1 \leq i \leq n\} \in \Omega(C_{2n+1})$, and if $u = v_{2i}$ (for some $i \neq 0$), then $x, y \notin S = \{v_2, v_4, \dots, v_{2i-2}\} \cup \{v_{2i+1}, v_{2i+3}, \dots, v_{2n+1}\} \in \Omega(C_{2n+1})$. Hence, we infer that S is stable in $C_{2n+1} + e_1 + e_2$, as well, in contradiction with $\alpha(C_{2n+1} + e_1 + e_2) < \alpha(C_{2n+1})$. Therefore, C_{2n+1} is α^{++} -stable. ■

Combining Corollary 3.12 and Proposition 3.11 we get the following extension of one Ravindra's theorem, [14], where he proved the first three equivalences.

Corollary 3.14 *For a tree T the following assertions are equivalent:*

- (i) T is well-covered;
- (ii) T has a perfect matching consisting of pendant edges;
- (iii) T is very well-covered;
- (iv) T is α^{++} -stable.

4 Conclusions

In this paper we keep investigating graphs whose stability number is invariant with respect to some natural operations on graphs. While in [10], [12] we were interested in measuring the influence of adding one edge to a graph, here we define a class of graphs whose stability number is unaffected by two edges addition.

Further we concentrate on König-Egerváry graphs, which is one of the most attractive generalizations of bipartite graphs. On the one hand, Proposition 3.11 claims that for girth ≥ 6 , α^{++} -stable König-Egerváry graphs and well-covered graphs are the same. On the other hand, Theorem 3.10 shows that an α^{++} -stable König-Egerváry graph contains no cycle on 4 vertices. It leaves an interesting open question concerning interconnections between well-covered graphs and α^{++} -stable König-Egerváry graphs of girth 3 or 5.

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